

# AUSLANDER-REITEN SEQUENCES ON SCHEMES

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ABSTRACT. Let  $X$  be a smooth projective scheme of dimension  $d \geq 1$  over the field  $k$ , and let  $\mathcal{C}$  be an indecomposable coherent sheaf on  $X$ . Then there is an Auslander-Reiten sequence in the category of quasi-coherent sheaves on  $X$ ,

$$0 \rightarrow (\Sigma^{d-1}\mathcal{C}) \otimes \omega \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \rightarrow 0.$$

Here  $\Sigma^{d-1}\mathcal{C}$  is the  $(d-1)$ 'st syzygy in a minimal injective resolution of  $\mathcal{C}$ , and  $\omega$  is the dualizing sheaf of  $X$ .

## 0. INTRODUCTION

This note shows that Auslander-Reiten sequences frequently exist in categories of quasi-coherent sheaves on schemes. More precisely, let  $X$  be a smooth projective scheme of dimension  $d \geq 1$  over the field  $k$ , and let  $\mathcal{C}$  be an indecomposable coherent sheaf on  $X$ . Then by theorem 3.2, there is an Auslander-Reiten sequence in the category of quasi-coherent sheaves on  $X$ ,

$$0 \rightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \rightarrow 0. \quad (1)$$

Moreover,  $\mathcal{A}$  can be computed: It is  $(\Sigma^{d-1}\mathcal{C}) \otimes \omega$ , where  $\Sigma^{d-1}\mathcal{C}$  is the  $(d-1)$ 'st syzygy in a minimal injective resolution of  $\mathcal{C}$  in the category of quasi-coherent sheaves, and  $\omega$  is the dualizing sheaf of  $X$ .

The sheaves  $\mathcal{A}$  and  $\mathcal{B}$  are not in general coherent, but only quasi-coherent. This is analogous to ring theory: If  $C$  is a finitely presented non-projective  $R$ -module with local endomorphism ring, then by [1, thm. 4] there is an Auslander-Reiten sequence in the category of all  $R$ -modules,

$$0 \rightarrow A \longrightarrow B \longrightarrow C \rightarrow 0,$$

but  $A$  and  $B$  are not in general finitely presented.

However, note that if  $X$  is a curve, then  $d = 1$ , and then  $\Sigma^{d-1}\mathcal{C}$  is just  $\mathcal{C}$  which is coherent, so in this case,  $\mathcal{A}$  and  $\mathcal{B}$  are coherent. So if  $X$  is a curve, then I recover the result known from [9] that the category of

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coherent sheaves on  $X$  has Auslander-Reiten sequences; see corollary 3.3.

The proof that the Auslander-Reiten sequence (1) exists is based on proposition 1.1 which uses another form of duality than the classical Auslander-Reiten formula to get Auslander-Reiten sequences. This may be of independent interest.

### 1. AUSLANDER-REITEN SEQUENCES AND DUALITY

**Proposition 1.1.** *Let  $\mathcal{A}$  be a  $k$ -linear abelian category with enough injectives, over the field  $k$ . Let  $\mathcal{C}$  be an object with local endomorphism ring. Let  $\mathcal{A}$  be another object for which there is a natural equivalence*

$$\mathrm{Hom}(\mathcal{C}, -)' \simeq \mathrm{Ext}^d(-, \mathcal{A}) \quad (2)$$

for some  $d \geq 1$ , where the prime denotes dualization with respect to  $k$ .

Then there is a short exact sequence

$$0 \rightarrow \Sigma^{d-1}\mathcal{A} \rightarrow \mathcal{B} \xrightarrow{b} \mathcal{C} \rightarrow 0$$

where  $b$  is right almost split. Here  $\Sigma^{d-1}\mathcal{A}$  is the  $(d-1)$ 'st syzygy in an injective resolution of  $\mathcal{A}$ .

*Proof.* Denote by  $J$  the Jacobson radical of the endomorphism ring  $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ , and pick a non-zero linear map  $\epsilon$  in  $\mathrm{Hom}(\mathcal{C}, \mathcal{C})'$  which vanishes on  $J$ . By the natural equivalence (2) the map  $\epsilon$  corresponds to a non-zero element  $e$  in  $\mathrm{Ext}^d(\mathcal{C}, \mathcal{A})$ .

Let  $0 \rightarrow \mathcal{A} \rightarrow \mathcal{I}^0 \rightarrow \dots \rightarrow \mathcal{I}^{d-1} \xrightarrow{i} \Sigma^d\mathcal{A} \rightarrow 0$  be terms number 0 to  $d-1$  of an injective resolution of  $\mathcal{A}$ , augmented by  $\mathcal{A}$  and the  $d$ 'th syzygy  $\Sigma^d\mathcal{A}$ .

Pick a morphism  $\mathcal{C} \xrightarrow{c} \Sigma^d\mathcal{A}$  which represents the element  $e$  in  $\mathrm{Ext}^d(\mathcal{C}, \mathcal{A})$ , and form the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Sigma^{d-1}\mathcal{A} & \longrightarrow & \mathcal{B} & \xrightarrow{b} & \mathcal{C} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow c \\ 0 & \longrightarrow & \Sigma^{d-1}\mathcal{A} & \longrightarrow & \mathcal{I}^{d-1} & \xrightarrow{i} & \Sigma^d\mathcal{A} \longrightarrow 0, \end{array}$$

where the lower row comes from the injective resolution of  $\mathcal{A}$ , and where the right hand square is formed by pull back; cf. [6, prop. VIII.4.2].

The morphism  $b$  is not split, for otherwise it would be easy to see that  $c$  factored through  $i$ , but then  $c$  would represent 0 in  $\mathrm{Ext}^d(\mathcal{C}, \mathcal{A})$ , contradicting that  $c$  represents  $e$  which is non-zero.

Now let  $\mathcal{M} \xrightarrow{m} \mathcal{C}$  be a morphism which is not a split epimorphism. Then  $\text{Hom}(\mathcal{C}, \mathcal{M}) \xrightarrow{\text{Hom}(\mathcal{C}, m)} \text{Hom}(\mathcal{C}, \mathcal{C})$  is not surjective, for  $\text{id}_{\mathcal{C}}$  is not in the image. Now,  $\text{Hom}(\mathcal{C}, m)$  is clearly a homomorphism of right-modules over the ring  $\text{Hom}(\mathcal{C}, \mathcal{C})$ . The target of this homomorphism is  $\text{Hom}(\mathcal{C}, \mathcal{C})$  itself which is a local ring. Since  $\text{Hom}(\mathcal{C}, m)$  is not surjective, its image must be contained in  $J$ , which is (among other things) the unique maximal proper right-submodule of  $\text{Hom}(\mathcal{C}, \mathcal{C})$ .

Hence the linear map  $\epsilon$  vanishes on the image of  $\text{Hom}(\mathcal{C}, m)$ , so  $\text{Hom}(\mathcal{C}, m)'$  sends  $\epsilon$  to 0. By the equivalence (2) this means that  $\text{Ext}^d(m, \mathcal{A})$  sends  $e$  to 0. Now,  $e$  is represented by  $\mathcal{C} \xrightarrow{c} \Sigma^d \mathcal{A}$ , so  $\text{Ext}^d(m, \mathcal{A})(e)$  is represented by  $\mathcal{M} \xrightarrow{cm} \Sigma^d \mathcal{A}$ , so  $\text{Ext}^d(m, \mathcal{A})(e) = 0$  implies that  $\mathcal{M} \xrightarrow{cm} \Sigma^d \mathcal{A}$  factors through  $\mathcal{J}^{d-1} \xrightarrow{i} \Sigma^d \mathcal{A}$ . So there is a commutative diagram

$$\begin{array}{ccc}
 \mathcal{M} & & \\
 \searrow & & \searrow m \\
 & \mathcal{B} & \xrightarrow{b} \mathcal{C} \\
 & \downarrow & \downarrow c \\
 & \mathcal{J}^{d-1} & \xrightarrow{i} \Sigma^d \mathcal{A}
 \end{array}$$

But the square is a pullback square, so the diagram can be completed with a morphism  $\mathcal{M} \longrightarrow \mathcal{B}$ , so  $m$  factors through  $b$ .  $\square$

## 2. A LEMMA ON ENDOMORPHISM RINGS

**Lemma 2.1.** *Let  $\mathcal{A}$  be an abelian category with a short exact sequence*

$$0 \rightarrow \mathcal{A} \xrightarrow{a} \mathcal{J} \xrightarrow{i} \Sigma \mathcal{A} \rightarrow 0,$$

*where  $\mathcal{A} \xrightarrow{a} \mathcal{J}$  is an injective envelope and where  $\text{Ext}^1(\mathcal{J}, \mathcal{A}) = 0$ .*

*If  $\mathcal{A}$  has local endomorphism ring, then so does  $\Sigma \mathcal{A}$ .*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{J} & \xrightarrow{i} & \Sigma\mathcal{A} \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \iota & & \downarrow \sigma \\
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{J} & \xrightarrow{i} & \Sigma\mathcal{A} \longrightarrow 0.
 \end{array}$$

It is clear that any morphism  $\alpha$  gives rise to such a diagram. However, so does any morphism  $\sigma$ , for when  $\sigma$  is given, consider  $\mathcal{J} \xrightarrow{\sigma i} \Sigma\mathcal{A}$  which represents an element in  $\text{Ext}^1(\mathcal{J}, \mathcal{A})$ . Since this Ext is 0, the morphism  $\mathcal{J} \xrightarrow{\sigma i} \Sigma\mathcal{A}$  must factor through  $\mathcal{J} \xrightarrow{i} \Sigma\mathcal{A}$ . This gives the morphism  $\mathcal{J} \xrightarrow{\iota} \mathcal{J}$ , and the morphism  $\mathcal{A} \xrightarrow{\alpha} \mathcal{A}$  and hence the diagram follows.

Observe that in the diagram, if  $\alpha$  is an isomorphism, then so is  $\iota$  because  $\mathcal{A} \xrightarrow{i} \mathcal{J}$  is an injective envelope, and hence, so is  $\sigma$ .

To show that  $\text{End}(\Sigma\mathcal{A})$  is local, I must show that if  $\sigma$  is a non-invertible element, then  $\text{id}_{\Sigma\mathcal{A}} - \sigma$  is invertible. That  $\sigma$  is non-invertible means that it is not an isomorphism. Embed  $\sigma$  in a diagram as above. Then  $\alpha$  is not an isomorphism, for otherwise  $\sigma$  would be an isomorphism, as observed above. So  $\alpha$  is a non-invertible element of  $\text{End}(\mathcal{A})$ . But then  $\text{id}_{\mathcal{A}} - \alpha$  is invertible because  $\text{End}(\mathcal{A})$  is local. That is,  $\text{id}_{\mathcal{A}} - \alpha$  is an isomorphism. But there is also a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{J} & \xrightarrow{i} & \Sigma\mathcal{A} \longrightarrow 0 \\
 & & \downarrow \text{id}_{\mathcal{A}} - \alpha & & \downarrow \text{id}_{\mathcal{J}} - \iota & & \downarrow \text{id}_{\Sigma\mathcal{A}} - \sigma \\
 0 & \longrightarrow & \mathcal{A} & \xrightarrow{a} & \mathcal{J} & \xrightarrow{i} & \Sigma\mathcal{A} \longrightarrow 0,
 \end{array}$$

and as observed above, when  $\text{id}_{\mathcal{A}} - \alpha$  is an isomorphism, so is  $\text{id}_{\Sigma\mathcal{A}} - \sigma$ . That is,  $\text{id}_{\Sigma\mathcal{A}} - \sigma$  is an invertible element of  $\text{End}(\Sigma\mathcal{A})$ .  $\square$

### 3. SCHEMES

The following lemma uses the dualizing sheaf of a projective scheme, as described for instance in [3, sec. III.7].

**Lemma 3.1.** *Let  $X$  be a projective scheme of dimension  $d$  with Gorenstein singularities, over the field  $k$ . Let  $\omega$  be the dualizing sheaf.*

*Let  $\mathcal{C}$  be a coherent sheaf which has a bounded resolution of locally free coherent sheaves. Then there are natural equivalences*

$$\text{Ext}_{\text{QCoh } X}^i(\mathcal{C}, -)' \simeq \text{Ext}_{\text{QCoh } X}^{d-i}(-, \mathcal{C} \otimes \omega),$$

where  $\mathbf{QCoh}(X)$  is the category of quasi-coherent sheaves on  $X$ .

*Proof.* Since  $X$  is projective over  $k$ , there is a projective morphism  $X \xrightarrow{f} \mathrm{Spec}(k)$ . This is certainly a separated morphism of quasi-compact separated schemes, so according to [7, exam. 4.2] the derived global section functor of  $X$ ,

$$R\Gamma : \mathbf{D}(\mathbf{QCoh} X) \longrightarrow \mathbf{D}(\mathbf{Mod} k),$$

(which equals the derived direct image functor  $Rf_*$ ) has a right-adjoint

$$f^! : \mathbf{D}(\mathbf{Mod} k) \longrightarrow \mathbf{D}(\mathbf{QCoh} X).$$

It is easy to see  $f^!k \cong \omega[d]$ ; cf. [7, rmk. 5.5]. Observe also that since the singularities of  $X$  are Gorenstein,  $\omega$  is an invertible sheaf.

Now consider the following sequence of natural isomorphisms which is taken from [5, sec. 4]. The object  $\mathcal{N}$  is in  $\mathbf{D}(\mathbf{QCoh} X)$ .

$$\begin{aligned} & \mathrm{Hom}_{\mathbf{D}(\mathbf{QCoh} X)}(\mathcal{C}, \mathcal{N})' \\ & \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{Mod} k)}(R\Gamma(R\mathcal{H}om_X(\mathcal{C}, \mathcal{N})), k) \\ & \cong \mathrm{Hom}_{\mathbf{D}(\mathbf{QCoh} X)}(R\mathcal{H}om_X(\mathcal{C}, \mathcal{N}), f^!k) \\ & \stackrel{(a)}{\cong} \mathrm{Hom}_{\mathbf{D}(\mathbf{QCoh} X)}(R\mathcal{H}om_X(\mathcal{C}, \mathcal{O}_X) \overset{\mathrm{L}}{\otimes}_X \mathcal{N}, f^!k) \\ & \stackrel{(b)}{\cong} \mathrm{Hom}_{\mathbf{D}(\mathbf{QCoh} X)}(\mathcal{N}, R\mathcal{H}om_X(R\mathcal{H}om_X(\mathcal{C}, \mathcal{O}_X), f^!k)) \\ & \stackrel{(c)}{\cong} \mathrm{Hom}_{\mathbf{D}(\mathbf{QCoh} X)}(\mathcal{N}, \mathcal{C} \overset{\mathrm{L}}{\otimes}_X f^!k). \end{aligned}$$

Here (b) is by adjointness between  $\overset{\mathrm{L}}{\otimes}$  and  $R\mathcal{H}om$ , while (a) and (c) hold because they clearly hold if  $\mathcal{C}$  is a locally free coherent sheaf, and therefore also hold for the given  $\mathcal{C}$  because it has a bounded resolution of locally free coherent sheaves, so is finitely built in  $\mathbf{D}(\mathbf{QCoh} X)$  from such sheaves.

Now,  $f^!k$  is  $\omega[d]$  and  $\omega$  is an invertible sheaf. Hence  $\mathcal{C} \overset{\mathrm{L}}{\otimes}_X f^!k$  is just  $(\mathcal{C} \otimes \omega)[d]$ . Inserting this along with  $\mathcal{N} = \mathcal{M}[i]$  with  $\mathcal{M}$  in  $\mathbf{QCoh}(X)$  gives natural isomorphisms

$$\mathrm{Ext}_{\mathbf{QCoh} X}^i(\mathcal{C}, \mathcal{M})' \cong \mathrm{Ext}_{\mathbf{QCoh} X}^{d-i}(\mathcal{M}, \mathcal{C} \otimes \omega),$$

proving the lemma.  $\square$

**Theorem 3.2.** *Let  $X$  be a smooth projective scheme of dimension  $d \geq 1$  over the field  $k$ . Let  $\omega$  be the dualizing sheaf.*

*Let  $\mathcal{C}$  be an indecomposable coherent sheaf. Then there is an Auslander-Reiten sequence in the category of quasi-coherent sheaves,*

$$0 \rightarrow (\Sigma^{d-1}\mathcal{C}) \otimes \omega \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \rightarrow 0.$$

*Here  $\Sigma^{d-1}\mathcal{C}$  is the  $(d-1)$ 'st syzygy in a minimal injective resolution of  $\mathcal{C}$  in the category of quasi-coherent sheaves.*

*Proof.* The proof will give slightly more than stated: Let  $X$  be a projective scheme over  $k$  of dimension  $d \geq 1$  with Gorenstein singularities, and let  $\mathcal{C}$  be an indecomposable coherent sheaf which has a bounded resolution of locally free coherent sheaves. Then I will prove that the indicated Auslander-Reiten sequence exists. (In the smooth case, each coherent sheaf has a resolution as required by [3, exer. III.6.5]).

The category of quasi-coherent sheaves  $\mathbf{QCoh}(X)$  is clearly  $k$ -linear, and by [4, lem. 3.1], it is a Grothendieck category, that is, a cocomplete abelian category with a generator and exact filtered colimits. Hence it has injective envelopes by [8, thm. 10.10], and in particular, it has enough injectives.

Also, the category of coherent sheaves is an abelian category so has split idempotents. It also has finite dimensional Hom sets, as follows e.g. from Serre finiteness, [3, thm. III.5.2(a)]. Hence the endomorphism ring of the indecomposable sheaf  $\mathcal{C}$  is local; cf. [10, p. 52].

Finally, if I set  $i = 0$  in lemma 3.1, then I get the natural equivalence

$$\mathrm{Hom}(\mathcal{C}, -)' \simeq \mathrm{Ext}^d(-, \mathcal{C} \otimes \omega).$$

This is equation (2) of proposition 1.1, with  $\mathcal{A} = \mathcal{C} \otimes \omega$ .

So all the conditions of proposition 1.1 are satisfied, and hence, there is a short exact sequence in  $\mathbf{QCoh}(X)$ ,

$$0 \rightarrow \Sigma^{d-1}(\mathcal{C} \otimes \omega) \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \rightarrow 0,$$

where  $b$  is right almost split. If I construct  $\Sigma^{d-1}(\mathcal{C} \otimes \omega)$  by means of a minimal injective resolution of  $\mathcal{C} \otimes \omega$ , then this sequence equals the one in the theorem because  $\omega$  is invertible. So to finish the proof, I must show that  $a$  is left almost split. Since  $b$  is right almost split, it is enough to show that  $\mathrm{End}(\Sigma^{d-1}(\mathcal{C} \otimes \omega))$  is a local ring, by classical Auslander-Reiten theory.

For this, note that the minimal injective resolution gives short exact sequences

$$0 \rightarrow \Sigma^\ell(\mathcal{C} \otimes \omega) \longrightarrow \mathcal{I}^\ell \longrightarrow \Sigma^{\ell+1}(\mathcal{C} \otimes \omega) \rightarrow 0$$

for  $\ell = 0, \dots, d-2$ , where  $\Sigma^\ell(\mathcal{C} \otimes \omega) \longrightarrow \mathcal{I}^\ell$  is an injective envelope. Hence, starting with the knowledge that  $\mathrm{End}(\mathcal{C} \otimes \omega) \cong \mathrm{End}(\mathcal{C})$  is local,

successive uses of lemma 2.1 will prove that all the rings

$$\mathrm{End}(\mathcal{C} \otimes \omega), \dots, \mathrm{End}(\Sigma^{d-1}(\mathcal{C} \otimes \omega))$$

are local, provided that I can show

$$\mathrm{Ext}^1(\mathcal{J}^\ell, \Sigma^\ell(\mathcal{C} \otimes \omega)) = 0$$

for  $\ell = 0, \dots, d-2$ , that is,

$$\mathrm{Ext}^{\ell+1}(\mathcal{J}^\ell, \mathcal{C} \otimes \omega) = 0 \quad \text{for } \ell = 0, \dots, d-2. \quad (3)$$

However, if  $\mathcal{J}$  is any injective, then lemma 3.1 gives

$$\mathrm{Ext}^j(\mathcal{J}, \mathcal{C} \otimes \omega) \cong \mathrm{Ext}^{d-j}(\mathcal{C}, \mathcal{J})' = 0$$

for  $j = 1, \dots, d-1$ , and this implies (3).  $\square$

Theorem 3.2 allows me to recover the following result known from [9].

**Corollary 3.3.** *Let  $X$  be a smooth projective curve over the field  $k$ . Then the category of coherent sheaves has right and left Auslander-Reiten sequences.*

Indeed, if  $\mathcal{C}$  and  $\mathcal{A}$  are indecomposable coherent sheaves, then there are Auslander-Reiten sequences of coherent sheaves,

$$0 \rightarrow \mathcal{C} \otimes \omega \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \rightarrow 0$$

and

$$0 \rightarrow \mathcal{A} \longrightarrow \mathcal{B}' \longrightarrow \mathcal{A} \otimes \omega^{-1} \rightarrow 0.$$

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## REFERENCES

- [1] M. Auslander, *A survey of existence theorems for almost split sequences*, pp. 81–89 in “Representations of Algebras” (Proceedings of the Durham symposium 1985), edited by P. Webb, London Math. Soc. Lecture Note Ser., Vol. 116, Cambridge University Press, Cambridge, 1986.
- [2] A. Beligiannis, *Auslander-Reiten triangles, Ziegler spectra and Gorenstein rings*, preprint (2003).
- [3] R. Hartshorne, “Algebraic Geometry”, Grad. Texts in Math., Vol. 52, Springer, Berlin, 1977.
- [4] L. Illusie, *Existence de résolutions globales*, pp. 160–221 in “Théorie des Intersections et Théorème de Riemann-Roch” (SGA 6), edited by P. Berthelot, A. Grothendieck, and L. Illusie, Lecture Notes in Math., Vol. 225, Springer, Berlin, 1971.

- [5] H. Krause, A note on Auslander-Reiten triangles and a theorem of Zimmermann, preprint (2003).
- [6] S. Mac Lane, “Categories for the Working Mathematician”, Grad. Texts in Math., Vol. 5, Springer, Berlin, 1971.
- [7] A. Neeman, *The Grothendieck duality theorem via Bousfield’s techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), 205–236.
- [8] N. Popescu, “Abelian Categories with Applications to Rings and Modules”, London Math. Soc. Monogr., Vol. 3, Academic Press, New York, 1973.
- [9] I. Reiten and M. Van den Bergh, *Noetherian hereditary abelian categories satisfying Serre duality*, J. Amer. Math. Soc. **15** (2002), 295–366.
- [10] C. M. Ringel, “Tame algebras and quadratic forms”, Lecture Notes in Math., Vol. 1099, Springer, Berlin, 1984.

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